Indeterminacy in Search Theory of Money: Bilateral vs. Multilateral Trades*

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Abstract

Several search models with divisible money share a property called real indeterminacy of stationary equilibrium. Although it is a striking theoretical result, the fundamental reason of the indeterminacy has not been fully understood yet. This paper finds that bilateral trade assumption common in search theory might cause the indeterminacy. I develop a price-posting model with two alternative trade procedures: a usual bilateral trade with random matching and also a multilateral trade. Then, while the equilibrium is indeterminate with the former setting, the latter multilateral trade assumption derives unique equilibrium. This paper also shows that similar indeterminacy and uniqueness arise even in Walrasian markets.

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1 Introduction

Several search models with divisible money show a property called real indeterminacy of stationary equilibria. Typically, the real indeterminacy means that, there exists a continuum of stationary equilibria where each equilibrium is not only different in nominal terms, rather in real variables such as consumption, labor, matching rate, etc. The indeterminacy of equilibrium *path* is found in many macroeconomic models as summarized in Benhabib and Farmer (1999), while the real indeterminacy of *stationary* equilibria is uncommon. However, interestingly, the real indeterminacy of stationary equilibria is derived in several search models with divisible money[1]. These papers study models with non-degenerated money holding distributions. Hence, the real indeterminacy of stationary equilibria in the literature implies that there exists a continuum of stationary money holding distributions.

The real indeterminacy has two undesirable features. First, it might create difficulty in solving search models of money with standard solution methods in macroeconomics such as guess and verify approach or computational method[2]. These techniques might find one or a

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[1] The list of papers is Green and Zhou (1998), Green and Zhou (2002), Kamiya and Sato (2004), Kamiya and Shimizu (2006), Kamiya and Shimizu (2007), Kamiya and Shimizu (2013), Kamiya and Shimizu (2011), Ishihara (2010), Matsui and Shimizu (2005), Sugaya (2008), Zhou (1999), and Zhou (1999)

[2] See, e.g., Molico (2006).

few equilibria, but there could exist many (possibly infinite) other equilibria. It is hard to obtain the global structure of equilibrium set. Second, the indeterminacy obstructs policy analyses on money search models. If each equilibrium has a different policy implication, researchers are unable to pick an equilibrium for evaluating its effect. To overcome these difficulties, the understanding about the indeterminacy is demanded. However, as far as the author knows, no paper clearly detects the cause of indeterminacy. There remains an important open question about the property of search theory of money.

This paper is an attempt to find out the logic of the indeterminacy and suggests a way to sustain unique equilibrium. I develop a model of price posting where each seller posts a unit price of consumption good, and each buyer chooses a submarket. After that, each buyer decides a purchase quantity, then a trade procedure clears the market. It resembles daily shoppings, e.g., a person observes a price of meat per wight in a supermarket and decides the quantity. On this setting, I consider two alternative trade procedures: a bilateral trade with random matching and a multilateral trade. The former is similar to standard competitive search model such as Moen (1997), and the latter is a variant of Shapley and Shubik (1977)'s market game. Then, the model with the bilateral trade exhibits the indeterminacy, but one with the multilateral trade shows unique equilibrium. The outcomes suggest that the bilateral trade might be the cause of the indeterminacy. The result can be understood intuitively. The bilateral trade assumption always equalizes the number of buyers and sellers. It makes one more identity in the system of equations to determine equilibrium money holding distribution. This identity makes one degree of freedom and derives a continuum of solutions.

In addition to the price posting market, the present paper also shows that a bilateral trade is possible to make indeterminacy even in Walrasian market. I construct a standard Walrasian market (trivially multilateral trade) model and a modified one incorporating a bilateral trade restriction. Then, similarly, the former derives unique equilibrium, while the latter exhibits the real indeterminacy. The result might answer the question why the real indeterminacy of stationary equilibrium is uncommon in macroeconomic models except search theory of money. It is because Walrasian market implicitly assumes multilateral trade.

Relation to the Literature First, this paper contributes to the debate about the reason of the real indeterminacy of stationary equilibria in search theory of money. Reacting to the initial indeterminacy result by Green and Zhou (1998), Wallace (1998) conjectures that the nominal nature of money is the cause of the indeterminacy. But, Zhou (2003) brings up a question because she shows that the indeterminacy exists even if commodity money is considered. There is another conjecture that indivisibility of good assumed in Green and Zhou (1998) is the source of indeterminacy. However, Kamiya and Shimizu (2006) and Ishihara (2010) derive the indeterminacy with divisible good. Kamiya and Shimizu (2006) succeed to uncover equilibrium conditions which obtain the real indeterminacy, but they do not detect what economic assumptions actually derive the conditions. Later, Kamiya and Shimizu (2013) study a centralized auction market and conjecture that the non-Walrasian price determination is the reason of the real indeterminacy. However, I cast doubt on the conjecture because my paper derives the similar indeterminacy even in Walrasian market. I will further investigate this point in Section 6. As far as I know, the present paper is the first one to provide a non-Walrasian market which derives unique equilibrium in the literature. Hence, this paper's conjecture has some validity compared to existing studies.

Secondly, this paper is related to the literature on monetary trade and real indeterminacy of stationary equilibria. Green and Zhou (1998), Green and Zhou (2002), Kamiya and Sato (2004), Kamiya and Shimizu (2006), Kamiya and Shimizu (2007), Kamiya and Shimizu (2011) and Ishihara (2010) consider search theories of money with non-degenerated distribution and show the real indeterminacy with variety of assumptions. There are some extensions such as market place choice by Matsui and Shimizu (2005), labor market by Sugaya (2008), and auction market by Kamiya and Shimizu (2013). These papers seem to share the same root of the indeterminacy. Interestingly, Jean et al. (2010) also show the real indeterminacy based on Lagos and Wright (2005). In addition, Kamiya et al. (2011) find the real indeterminacy in cash-in-advance Walrasian economy. These papers derive degenerated money holding distributions; hence, the logic of the indeterminacy is different from the papers with non-generated distributions. My paper finds the real indeterminacy of stationary money holding distribution in a type of price-posting or competitive search model. In this sense, the present paper adds another example to the literature on the real indeterminacy.

Finally, this paper also has connections to the literature on monetary trade with non-random search. With indivisible money, Corbae et al. (2003) consider a type of coalition formation game, Howitt (2005) studies the appearance of trading place, and Julien et al. (2008) analyze an auction market. Based on Lagos and Wright (2005), Lagos and Rocheteau (2005) and Rocheteau and Wright (2005) build directed market models, and Galenianos and Kircher (2008) consider an auction market. Recently, Menzio et al. (2013) find an equilibrium with non-degenerated money holding distribution with a block recursive structure[3]. This paper's multilateral trade model is another example of non-random search theory. This paper succeeds to derive analytical result with simple non-degenerate money holding distribution.

Organization of the paper In Section 2, I propose a simple example to provide intuition how a bilateral trade assumption makes equilibrium indeterminate. In Section 3, the environment of the economy and equilibrium concept are introduced. In Section 4, I consider a bilateral trade procedure and derive the real indeterminacy. In Section 5, I solve a model with

[3]Interestingly, Menzio et al. (2013) derive the unique equilibrium. Their model succeeds to escape from the indeterminacy probably because (i) Walrasian labor market is assumed, (ii) agents do not alternate between buyers and sellers unlike typical money search models, (iii) only the space of continuous value functions is considered, while the real indeterminacy often held with step functions in the literature.

a multilateral trade and show equilibrium uniqueness. Section 6 considers Walrasian market with and without a bilateral trade procedure. I show that the difference in trade procedure is still important in Walrasian market. In Section 7, I analyze an abstract structure of the real indeterminacy in my models. The conclusion is in Section 8.

2 A simple example

To derive an intuition about my main result, This section proposes a simple example. There is one market which is divided into buyer and seller sides. Suppose that there are four homogeneous agents. Every period, each agent enters either side of the market. The total money supply in this economy is \$300. Agents holding positive amount of money must go to the buyer side and spend the whole amount. If agents have no money, they must enter the seller side and wait for a chance to receive money from agents in the buyer side. To concentrate on money circulation, ignore the real side such as production and consumption, and suppose that the agents just follow the above rules instead of incentives. The agents' optimal strategies will be considered in the following sections.

Under the setting, I propose two types of trade procedures.

1. A bilateral trade with random matching: a random draw select one agent from each side. The buyer side's agent gives the whole amount of money to the seller side agent.

2. A multilateral trade: all agents in the buyer side give all money. The money is distributed equally to agents in the seller side.

Suppose the first procedure. It follows the common feature of search theory of money. The bilateral trade assumption derives multiple stationary money holding distributions.

- One agent has \$300, and the others have nothing. Each period, the only one agent in the buyer side proffers \$300 to an agent in the seller side. Hence, the distribution is stationary.

- Two agents equally have \$150, and the other two have nothing. Each period, one agent from the buyer side gives \$150 to one agent in the seller side.

- Three agents equally have \$100, and the other has nothing. Each period, one agent from the buyer side donates \$100 to the agent in the seller side.

Therefore, the bilateral trade procedure exhibits the indeterminacy of stationary money holding distributions[4].

Next, suppose the second trade procedure. There exists only one stationary money holding distribution.

[4]There are only three distributions because the number of agents is four. If there is a continuum of agents, there exists a continuum of money holding distributions.

- Two agents equally have \$150, and the other two have nothing. Each agent from the buyer side gives \$150 to each agent in the seller side; hence, the distribution is stationary.

What does happen if one agent has \$300? Then, the \$300 is equally divided into three agents in the seller side. In the next period, the three agents will have \$100 each. Obviously, the money holding distribution is not stationary[5].

The bilateral trade procedure restricts the number of agents who can trade. It clearly helps to sustain multiple stationary money holding distributions. The bilateral trade forces to equalize the number of buyers and sellers. Then, whatever the shape of the distribution is, the procedure just exchange the same population between a point with another one on the support of the distribution. This hidden effect in search theory makes equilibrium indeterminate. Given the intuition, the rest of the paper shows the similar result in a more rigorous way.

3 Environment and Equilibrium Concept

In this section, I introduce a price posting model and its equilibrium concept. This section concentrates on defining the environment, and bilateral and multilateral trade procedures will be established in the following sections.

3.1 Environment

Time is discrete and infinite. There are two types of goods: consumption good and money. The consumption good is divisible and non-storable. Money is divisible and storable. For simplicity, I assume that money supply is constant M through all periods.

There exists a unit measure of homogeneous agents. Each agent has an index $i \in [0,1]$. An agent obtains utility uq when she consumes q units of the consumption good. In adition, each agent can produce the consumption good with a labor disutility function:

$$C(q) = \begin{cases} 0 & \text{if } q = 0 \\ c & \text{if } q \in (0,1] \\ +\infty & \text{if } q > 0 \end{cases}.$$

This means that each seller produces at most one unit of the consumption good with a constant disutility c. The function induces a strong incentive for sellers to cut prices and sell more because the marginal cost is zero for $q \in (0,1]$. It ensures the Bertrand-Edgeworth competition among sellers and let the equilibrium analysis simple. To enhance monetary trade, it is assumed that each agent can either only consume or only produce in each period[6]. Each agent discounts the

[5]There also exist cyclical money holding distributions. But, these are not robust to a small perturbation. See Section 5.5 for the detail.

[6]In addition, to avoid double-coincidence of wants, I assume a conventional trick in the literature. There are $K \geq 3$ types of agents, and the population of each type is $1/K$. A type k agent only consumes the good produced by type $k-1$. Since this economy deals with a price posting procedure, a type k buyer can go to a type $k-1$ seller's submarket. Hence, I can simply ignore the types. Furthermore, in order to get rid of repeated game type non-monetary equilibrium, I suppose that there is no memory, and the set of agents is continuum. See Kocherlakota (1998) and Araujo (2004) for the detail.

future with a discount factor β. Let m denote amount of money holding of an agent. This is the individual state variable.

In each period, the market proceeds as follows.

- Stage 1: each agent decides whether to become a seller or a buyer. It is assumed that an agent holding no money is restricted to choose to be a buyer[7].
- Stage 2: each seller posts monetary price of consumption good per unit p. Let $\mathcal{P}$ be the distribution of posted prices and $\underline{\mathcal{P}}$ denote its support. The sellers posting the same price p form Submarket p.
- Stage 3: each buyer chooses a submarket from $\underline{\mathcal{P}}$. They can observe the measure of sellers in each submarket[8].
- Stage 4: each buyer decides a purchase quantity given the price p. In Submarket p, a trade procedure which will be introduced later decides the quantity of trade.

This paper will deal with two types of procedures in Stage 4. I will propose a bilateral trade with search friction in Section 4 and a multilateral trade in Section 5.

3.2 Equilibrium Concept

This paper uses a game-theoretical concept. Let x_j be an agent's action, and X_j denote the set of available actions in Stage $j = 1, 2, 3, 4$. Then, $X_1, \ldots, X_4$ are defined as follows:

- $X_1 \equiv \{\text{SELLER, BUYER}\}$,
- $X_2 \equiv \mathbb{R}_+$ if $x_1 = \text{SELLER}$, otherwise $X_2 \equiv \emptyset$. The support of offered prices $\underline{\mathcal{P}}$ is determined by all sellers' decisions $x_2 \in X_2$.
- $X_3 \equiv \underline{\mathcal{P}}$ if $x_1 = \text{BUYER}$, otherwise $X_3 \equiv \emptyset$
- If $x_1 = \text{BUYER}$ and the agent succeed to match with a seller[9], $X_4 \equiv \mathbb{R}_+$, otherwise $X_4 \equiv \emptyset$.

Define $X \equiv X_1 \times \cdots \times X_4$, and $x(m) \in X$ be a strategy. A strategy $x(m)$ depends on the money holding m at the beginning of a period. Define $\Omega(x)$ as the stationary distribution of strategies. In addition, let λ denote the distirbution of money holdings across agents and $V(m)$ be the value function of an agent holding m units of money at the beginning of a period.

This paper's equilibrium concept follows the standard equilibrium definition in the literature since Green and Zhou (1998).

[7]This condition is satisfied endogeneously, but I assume it to simplify the proof.

[8]As in Footnote 6, the submarkets are also characterized by types $k = 1, \ldots, K$. A type k buyer can choose a submarket formed by type $k-1$ sellers.

[9]I will also impose budget constraint in each trade procedure.

Definition 1. *A stationary monetary equilibrium consists of Ω, λ and V which satisfy (i) λ is stationary under Ω, (ii) V is induced by Ω, (iii) Given λ and V, Ω consists subgame perfect equilibrium from Stage 1 to Stage 4, and Ω is stationary, (iv) V is weakly increasing in m, and for all m, there exists $m' > m$ such that $V(m) < V(m')$.*

The last condition assumes that money has at least some positive value. Note that this condition includes step value functions derived in several papers[10]. For common notations, define B_p and S_p be the set of buyers and sellers in Submarket p in Stage 4. Define also $B \equiv \bigcup_p B_p$ and $S \equiv \bigcup_p S_p$ be the set of all buyers and all sellers respectively. A function $h(Z)$ denotes the measure of agents in a set Z, e.g., $h(B_p)$ is the measure of buyers in Submarket p.

4 Random Search with Bilateral Trade

This section builds a type of competitive search model, i.e., Stage 4 consists of bilateral trade with search friction. Here, I will derive only a set of equilibria rather than all possible equilibria. But it is sufficient to show the existence of the real indeterminacy because the equilibrium set contains a continuum of stationary equilibria.

4.1 Environment at Stage 4

In Submarket p, $\min\{h(S_p), h(B_p)\}$ buyers and sellers are matched randomly[11]. After a buyer i succeeds to make a match with a seller, she decides a purchase quantity q_i under the resource constraint $q_i \leq 1$ and the budget constraint $pq_i \leq m$. The buyer freely determines the purchase quantity q_i given the constraints. If the matching is failed, agents do nothing.

4.2 Equilibrium Strategy

In this subsection, I first guess a set of equilibrium money holding distributions λ and value funcions V. For each guess, a subgame perfect equilibrium with a strategy x is derived in the next subsection. I will show that x derives the guess λ and V in order to verify the equilibrum. Because each guess has one equilibrium, it means that there exists an inderminacy.

In short, each equilibrium is summarized as:

- The money holding distribution λ is degenerated at two points: H_0 agents hold no money, and H_1 agents hold M/H_1 units. $H_1 > H_0$ is assumed,
- There exists a cutoff $\bar{m}$ such that an agent becomes a seller if $m < \bar{m}$ and a buyer if $m \geq \bar{m}$,

[10] The step value functions are derived in Green and Zhou (1998), Green and Zhou (2002), Kamiya and Sato (2004), Kamiya and Shimizu (2006), Kamiya and Shimizu (2007), Kamiya and Shimizu (2011), Ishihara (2010), Matsui and Shimizu (2005), Sugaya (2008), Zhou (1999), and Zhou (2003). A notable exception is Kamiya and Shimizu (2011), who derive a continuum value function.

[11] Acemoglu and Shimer (1999) call this matching function "frictionless." I assume it in order to compare the competitive search model with the multilateral matching model clearly. Alternatively, it can be generalized to the standard constant returns to scale matching function.

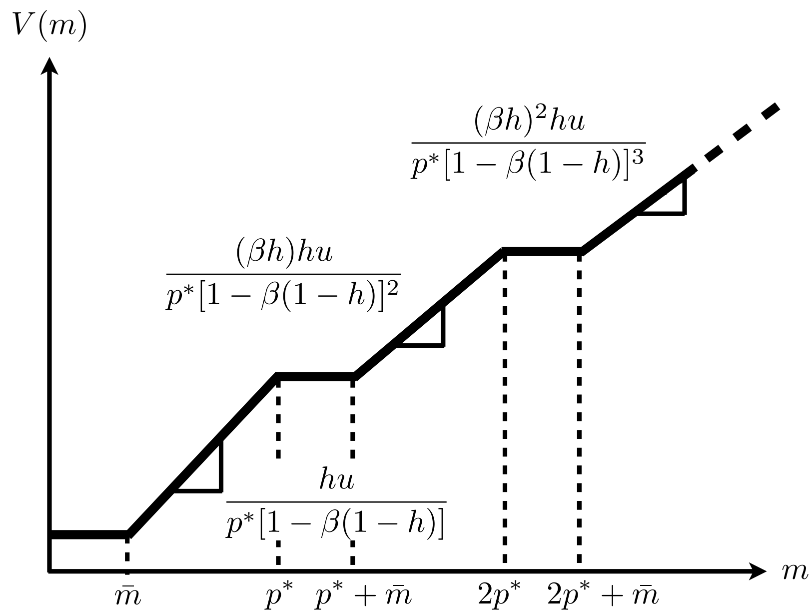

Figure 1: Value function in the bilateral trade case

- Because $\bar{m} < M/H_1$ is satisfied, H_0 agents choose to be sellers, and H_1 agents become buyers. The market makes $H_0 = \min\{H_0, H_1\}$ matches,
- All sellers offer the same price p^*,
- All buyer spends money as possible as they can, i.e., spend m if $m < p^*$ and p^* if $m \geq p^*$ because $q \leq 1$.

Note that the equilibrium is indeterminate because any distribution with $H_1 > H_0$ can be derived in an equilibrium. I will derive each equilibrium above from the following guess.

Equilibrium Guess 1. *First, the money holding distribution λ is degenerated at two points: H_0 agents hold no money, and H_1 agents hold M/H_1 units, where $H_0 < H_1$ and $H_0 + H_1 = 1$. Define $h \equiv H_0/H_1$. Second, the value function $V(m)$ is depicted as in Figure 1. To define formally, let $p^* = M/H_1 = \left(\frac{h}{1+h}\right)M$. For each m, two numbers $a \in \mathbb{N}$ and $b \in \mathbb{R}$ are uniquelly defined so that $m = ap^* + b$ where $b < p^*$. Then,*

$$V(m) = \frac{u}{\beta}\sum_{i=1}^{a}\left(\frac{\beta h}{1-\beta(1-h)}\right)^i + \left(\frac{\beta h}{1-\beta(1-h)}\right)^a \left[V(0) + \max\{b-\bar{m}, 0\} \cdot \left(\frac{hu}{p^*[1-\beta(1-h)]}\right)\right], \quad (1)$$

where

$$V(0) = \frac{\beta hu - (1-\beta+\beta h)c}{(1-\beta)(1+\beta h)} > 0, \tag{2}$$

$$\bar{m} = \frac{(1-\beta)p^*}{hu} V(0) = \frac{p^*[\beta hu - (1-\beta+\beta h)c]}{hu(1+\beta h)}. \tag{3}$$

The value function seems complicated, but I can simply show that it reflects the equilibrium behavior. The equilibrium price is p^*. First, consider a case that an agent holds $m \le p^*$ unit of money. In this case, $a = 0$, $b = m$ and

$$V(m) = V(0) + \max\{m - \bar{m}, 0\} \cdot \frac{u}{\beta p^*} \left(\frac{\beta h}{1-\beta(1-h)} \right) \tag{4}$$

There is a cutoff $\bar{m}$. If $m \le \bar{m}$, an agent becomes a seller. Note that this m units of money is never used on the equilibrium path, because the agent will earn p^* and later will be a seller and use the same p^* units of money. Therefore, $V(m) = V(0)$ for all $m \le \bar{m}$. If $m > \bar{m}$, the value function (1) can be rewritten as

$$V(m) = h \left[\frac{m}{p^*} u + \beta V(0) \right] + \beta(1-h)V(m). \tag{5}$$

The agent becomes a buyer. She can meet a seller with probability $h = H_0/H_1$ by the bilateral trade friction. If the buyer succeeds in matching, she spends all money and derive utility $(m/p^*)u$ in this period. Since she has nothing next period, the continuation utility is $\beta V(0)$. The term $\beta(1-h)V(m)$ shows a case that she fails to meet a seller.

Second, think about a case where $m > p^*$. The value function (1) can be similarly rewritten as

$$V(m) = h\left[u + \beta V(m - p^*)\right] + \beta(1-h)V(m), \tag{6}$$

This means that a buyer spends p^* unit of money and consume 1 unit of good if she meets a seller. The buyer is unable to use more than p^* because productions are limited to $q \le 1$.

4.3 Equilibrium

In the next proposition, I verify that Equilibrium Guess 1 actually consists a continuum of equilibria. The proof is in Appnendix.

Proposition 1. *For all $H_1 \in (1/2,\ 1)$, there exists an equilibrium such that the money holding distribution is degenerated at* 0 *and M/H_1.*

There is a continuum of equilibria under $H_1 > H_0$. In each equilibrium, H_0 agents hold no money and choose to be sellers, and H_1 have M/H_1 units of money and become buyers. Each period, $H_0 = \min\{H_0, H_1\}$ agents make matches and trade. By Proposition 1, a corollary immediately follows.

Corollary 1. *The indeterminacy is real.*

It implies that the real variables such as number of trades and economic welfare vary among equilibria as well as the nominal price. Each period, H_0 buyers derive utility u and H_0 sellers cost c. Therefore, the discounted welfare is $H_0(u-c)/(1-\beta)$. This is strictly increasing in H_0, which is indeterminate. Thus, the economic welfare is also indeterminate.

The result itself is unsurprising. While I consider a type of competitive search model, the main feature is not significantlly different from the random search literature since Green and Zhou (1998). In the next section, I will show that the indeterminacy vanishes if the trade pricedure is changed from bilateral to multilateral.

5 Multilateral trade

In this section, I will propose a mulilateral trade in Stage 4 and derive the unique equilibrium.

5.1 Environment in Stage 4

As defined above, let B_p and S_p be the set of buyers and sellers in Submarket p, and $h(B_p)$ and $h(S_P)$ be their measures, respectively.

In Stage 4, buyers offer their desired demand. Let q_i be the offer by buyer i. The aggregate demand can be written as $Q_p = \int_{B_p} q_i di$. The multilateral trade market is cleared as

- each seller sells $\min\left\{1, \dfrac{Q_p}{h(S_p)}\right\}$ units,
- buyer i purchases $q_i \cdot \min\left\{1, \dfrac{h(S_p)}{Q_p}\right\}$ units.

In Submarket p, the maximum supply is $h(S_p)$ becuase each seller can produce at most one unit. In case of excess demand $Q_p > h(S_p)$, the good is distributed according to offers q_i. All sellers produce one unit and buyer i acquires $q_i \cdot h(S_p)/Q_p$ units. In case of excess suply $Q_p < h(S_p)$, each buyer can acquire q_i unit. The amount of selling is limited so that each seller provides the same amount $Q_p/h(S_p) < 1$. Note that this quantity adjustment procedure is related to the market game defined by Shapley and Shubik (1977). In the market game, the nominal price per unit is determined by the demand/supply ratio $Q_p/h(S_p)$. By contrast, the price per unit is predetermined by the seller side price posting in my model. The demand/supply ratio $Q_p/h(S_p)$ is only used to determine allocation[12].

I consider the following budget constraint.

$$pq \cdot \min\left\{1, \frac{h(S_p)}{Q_p}\right\} \leq m. \tag{7}$$

[12]In a market game, a good is never allocated to a person who derives no utility from it, e.g., a seller in my model. But, my model allows the possibility that there remains good left unsold.

The constraint means that the buyer's payment on the equilibrium must not exceed money holding[13].

In addition to that, the model needs the following special rule because there might not exist equilibrium in each submarket.

Assumption 1. *If there does not exist Nash equilibrium in Submarket p, i.e., all buyers $i \in B_p$ post $q_i \to \infty$, the market decides a cutoff z where*

$$\int_{i \le z, i \in B_p} \left(\frac{m_i}{p} \right) di = h(S_p).$$

The purchase amount is exogeneously determined. It is m_i/p if $i \le z$ and 0 otherwise. Each seller selles one unit of good. This case must not happen on the equilibrium path.

The amount z is determined so that all buyers with indices $i \le z$ spend whole money holdings. I suppose that this case must not come up on the equilibrium path because Assumption 1 is intended only to eliminate trivial off-paths.

5.2 Properties of Equilibrium

In this subsection, to prepare to show the uniqueness, I will derive properties or necessary conditions that equilibrium must satisfy under the definition of stationary monetary equilibrium and the model's environment. If the uniqueness is shown in the set of candidates of equilibria that satisfy the properties, it implies the global uniqueness.

Property 1. *Each buyer can buy any amount in any submarket within her budget.*

In this economy, q_i is not the quantity which the buyer i wants to purchase. By the rule, it is defined as the maximal amount the buyer i acquires. Under the excess supply, the buyer i purchases q_i units, but if the demand exceeds supply, she obtains less than q_i. Hence, in the excess demand case, the buyer i has an incentive to tell q_i that is larger than the amount she wants. Like that, given other buyers' posts, the buyer i can purchase any amount of goods by posting larger amount within her budget[14].

Given Property 1, the following two properties are immediately dervied.

Property 2. *Buyers spend whole money holding.*

The price distribution is stationary on the equilibrium. By Proerty 1, buyers can choose the best market and spend the amount they want in any period. Since the utility function is linear and future consumption is discounted, there is no incentive to save money. Buyers spend all money at one time.

[13] It depends on a market outcome $h(S_p)/Q_p$; hence, a rational expectation is assumed. If a buyer could not predict the equilibrium, it would be $pq \le m$.

[14] Note that this property is just a trick to eliminate irrelevant off-paths. It does not happen on the equilibrium path as I will show later.

Property 3. *The trade occurs only in one market.*

Suppose that buyer i with m units of money selects market p. By Property 2, a buyer's discounted utility is

$$u\frac{m}{p} + \beta V(0).$$

This is strictly decreasing in p. Therefore, each buyer strictly prefers the lowest price market. No buyer comes to others. By Property 1, buyers always buy the amount they want in the lowest price market.

By Property 3, the model has only single-price equilibrium. The next property establishes the value function's shape and the equilibrium behavior in Stage 1.

Property 4. *There exists $\bar{m}$ such that $V(m)$ is linear and strictly increasing in $[\bar{m}, +\infty)$ with a slope of u/p. An agent holding less tham $\bar{m}$ becomes a seller, and more than or equal to $\bar{m}$ chooses to be a buyer in Stage 1.*

Proof. Appendix. □

The result is interpreted as follows. If an agent has plenty of money m, she will spend whole amount by Property 2 given the single price p by Property 3. The discounted utility is $(um/p) + \beta V(0)$, which is linear in m with a slope of u/p. If she has only small amount of money, she becomes a seller today instead of spending it. For a moneyless agent, being a buyer derives only small utility, and it loses a chance to earn money.

In the next subsection, I will use these properties to derive the unique equilibrium.

5.3 Equilibrium

In this subsection, I solve the model backward and derive the unique equilibrium.

Stage 4

Consider the decision of a buyer who holds m units of money in Submarket p. The buyer solves the problem given strategies of sellers and other buyers, i.e., $h(S_p)$ and Q_p. The problem is

$$\max_{q \in \mathbb{R}_+} \; u \cdot q \cdot \min\left\{1, \frac{h(S_p)}{Q_p}\right\} + \beta V\left(m - pq \cdot \min\left\{1, \frac{h(S_p)}{Q_p}\right\}\right) \tag{8}$$

$$\text{s.t.} \quad pq \cdot \min\left\{1, \frac{h(S_p)}{Q_p}\right\} \leq m.$$

By Property 3, there is only one price. By Property 2, the buyer spends whole amount of money. Therefore, if equilibrium exists, the optimal solution must be

$$q^* = \frac{m}{p \cdot \min\{1, h(S_p)/Q_p\}}. \tag{9}$$

The solution of the optimization depends on the shape of $V(m)$. I will verify that the above q^* is actually the optimum in Proposition 3.

Stage 3

By Property 3, the price is degenerated at one point on the equilibrium. Then, all buyers choose the market trivially. To verify the equilibrium, I consider an off-path that one seller deviates from the single price equilibrium. This means that the set of offered price $\mathcal{P}$ is degenerated at two prices p' and p, where $S_{p'} \neq \emptyset$ and its measure is $h(S_{p'}) = 0$. First consider a case where $p' > p$.

Lemma 1. *If $p' > p$, no buyer chooses p' on the equilibrium.*

Proof. Since the single price p is on the equilibrium path and $h(S_p') = 0$, the case of Assumption 1 is not applied. By the argument in Property 3, buyers have no incenive to choose p'. □

Next, consider a case that $p' < p$. In this case, I need to solve a situation that a measure 0 seller might meet a positive measure of buyers. To formulate the situation, I first derive a result in a case that $h(S_{p'}) = \bar{h} > 0$, and next assume that the result continuously holds if $\bar{h} \to 0$.

Lemma 2. *If $p' < p$, each seller $i \in S_{p'}$ sells one unit of good.*

Proof. First assume that $h(S_{p'}) = \bar{h} > 0$, where $\bar{h}$ is sufficientlly small. Since $p' < p$, all buyers have incentive to enter price p' market and spend all money. Since $h(S_{p'})$ is sufficiently small, the demand excesses supply, and buyers have incentive to submit q_i which is higher than they want. Therefore, $q_i \to \infty$ for all $i \in B_{p'}$. Then, Assumption 1 is applied. Because this result is predicted by buyers, only buyers with $i \leq z$ where $\int_{i \leq z, i \in B} (m_i/p) di = h(S_{p'})$ enters Submarket p' and others enter Submarket p. Hence, by Assumption 1, each seller sells one unit of good in Submarket p'. By assumption, the result still holds when $\bar{h} \to 0$. □

Stage 2

The price is degeneraed at one point. In this market, there are the set of sellers S and the set of buyers B. The next lemma derives that the equilibrium price p^* is determined so that sellers acquire whole amount of money held by buyers.

Lemma 3. *The equilibrium price p^* satisfies*

$$p^* = \frac{1}{h(S)} \int_{i \in B} m_i di.$$

Proof. Let the equilibrium price is degenerated at $p > p^*$. Then, each seller sales $p^*/p < 1$ units of goods and acquire p^* units of money. If a seller deviates and choose the price $p - \varepsilon$, by Lemma 2, the seller can sell 1 unit of the consumption good. The revenue will be $p - \varepsilon$; hence, there exits sufficeintly small $\varepsilon > 0$ such that sellers make more profit.

Next consider that the price is degenerated at $p < p^*$. In this case, $h(S) < \int_B (m_i/p)di$ is satisfied. By Stage 4 analysis, all buyers post higher quantity so that they can use all money holdings. Then, the posted quantity will goes to ∞. It does not hold equilibrium by Assumption 1. □

Then, each seller trades one unit of consumption good with $(1/h(S)) \int_B m_i di$ unit of money. All buyers will hold nothing in the next period.

Stage 1

By Property 4, agents holding less than $\bar{m}$ choose to be sellers, and more than or equal to $\bar{m}$ become buyers. I will derive the actual value of $\bar{m}$ and uniquenss of the equilibrium in the next subsection.

5.4 Unique Equilibrium

By the analysis above, buyers use all money and sellers acquire p^*. Therefore, the equilibrium stationary money holding distribution must be degenerated at 0 and p^*. Agents holding p^* unit of money become buyers and spend all, and agents with no money become sellers[15]. Let each population be H_1 and H_0 respectively.

Proposition 2. *If the equilibrium money holding distribution exists, it is degenerated at* 0 *and* $p^* = 2M$ *where both populations are* $1/2$.

Proof. Assume H_1 agents hold M/H_1 and $H_1 \neq 1/2$. In the next period, H_0 sellers acquire M/H_0. Since $M/H_1 \neq M/H_0$, it does not consist stationary distribution. Therefore, the unique equilibrium distribution is $H_0 = H_1 = 1/2$. □

Proposition 2 is the main point in this section. It means that if equilibrium exists, the equilibrium money holding distribution is unique. To complete the proof, I still need to show $p^* = 2M > \bar{m}$ and the optimality of (9).

Proposition 3. *If* $\beta u \geq c \geq (\beta + \beta^2 - 1)u$, *there exists the unique equilibrum.*

Proof. Appendix. □

In this section, I solve the model and find the unique equilibrium started from the properties. Because these properties are necessary conditions, the equilibrium is globally unique.

[15]Of course $p^* > \bar{m}$ is required. It will be considered in Proposition 2.

The equilibrium is summarized as: (i) half agents have nothing and choose to be sellers, (ii) the other half hold $2M$ units of money and become buyers, (iii) all sellers and all buyers exchange one unit of consumption good with $2M$ units of money, (iv) all buyers become sellers and all sellers turn to buyers next period.

5.5 Cyclical equilibrium

The model has established the uniqueness of the stationary equilibrium. But, there might also exist a continuum of cyclical equilibria. For example, suppose two numbers $A, B \in \mathbb{R}_{++}$ such that $A > B$ and $A + B = 1$. Then, there actually exists the following cyclical equilibrium.

- A agents hold M/A units of money, and other B agents holds nothing at even priods.
- B agents hold M/B units of money, and other A agents holds nothing at odd priods.

However, I can show the stationary distribution with $A = B = 1/2$ is the only robust one to a small perturbation.

Definition 2. *A small perturbation is a change of the model where $\varepsilon > 0$ measure of agents is selected randomly before Stage 1, and they must stay autarky at the period.*

For any sufficietnly small $\varepsilon > 0$, the cyclical equilibrium is not robust to the small perturbation. Suppose that A agents hold nothing in period t. Then, in $t+1$, $\varepsilon A + (1-\varepsilon)B$ agents hold no money. In $t+2$, the number becomes $\varepsilon[\varepsilon A + (1-\varepsilon)B] + (1-\varepsilon)[\varepsilon B + (1-\varepsilon)A]$. It is no more cyclical, and converges to $A = B = 1/2$. In this sense, $A = B = 1/2$ is the only robust distribution.

Note that this small perturbation is unable to select an equilibrium from the indeterminacy model with the bilateral trade in Section 4. Assume again that A agents hold no money in period t, and $A < B$. In this period, $(1-\varepsilon)A = \min\{(1-\varepsilon)A,\ (1-\varepsilon)B\}$ sellers acquire M/B units of money. At the same time, $(1-\varepsilon)A$ buyers spend all money holdings. Therefore, $(1-\varepsilon)A$ agents acquire money, εA agents stay autarky, and $(1-\varepsilon)A$ lose money. Hence, there will be A agents who hold nothing in $t+1$ again. The statinary equilibrium holds with any $0 < A < B$.

6 Walrasian market

In this section, I show that the difference in trade procedures is still important in Walrasian market. First, a Walrasian market model with monetary trade is considered. By definition, Walrasian market is a type of multilateral trade economy. It is shown that there exists unique equilibrium. Next, I consider a Walrasian market model with a bilateral trade restriction, i.e., the number of buyers and sellers must be equalized. Then, the model shows the real indeterminacy as in the model in Section 5. Bilateral trade can cause the real indeterminacy even in Walrasian market. The result implies that other proerties of Walrasian market, such as demand-suply equality or the way of price determination, have no effect to the uniqueness of equilibrium.

6.1 Model

Consider a unit measure of continuum of agents who have the same preference as in the price-posting model. Similarly, there are consumption good and fixed amount of money M. There is one competitive market which has two sides: seller and buyer. As the trading post model proposed by Hayashi et al. (1996), this assumption makes incentive for monetary trade in Walrasiam market. Each agent must decide which side to enter. I consider two cases about the central market: no restriction, and a bilateral trade.

6.2 No Restriction

I first solve the stationary equilibrium in the model without any restriction. Given that the good price is p and stationary, an agent solves

$$V(m) = \max\{V_s(m), V_b(m)\},$$

$$V_s(m) = \max_{q\in[0,1]} \; -c + \beta V(m + pq),$$

$$V_b(m) = \max_{q\in[0,m/p]} \; u \cdot \frac{q}{p} + \beta V(m - pq),$$

where V is the ex-ante discounted utility, V_s is the ex-post value if the agent chooses seller side, and V_b is the value of the buyer side. Since the utility is linear, the buyer has no incentive to save money; hence, $V_b(m) = um/p + \beta V(0)$. Since the cost of production is fixed, $q = 1$ is always an optimal choice for sellers. Therefore,

$$V(m) = \max\left\{-c + \beta V(m + p),\ \frac{um}{p} + \beta V(0)\right\}.$$

The solution is

$$V(m) = \begin{cases} \frac{\beta u - c}{1-\beta^2} & \text{if } m \le \bar{m}, \\ \frac{um}{p} + \beta\left(\frac{\beta u - c}{1-\beta^2}\right) & \text{if } m \ge \bar{m}, \end{cases}$$

where $\bar{m} = \frac{p(\beta u - c)}{u(1+\beta)}$. The optimal behavior is choosing to be a seller if $m \le \bar{m}$ and a buyer otherwise.

The equilibrium pattern of spending is the following. First, all buyers spend whole money holding. Next, they become sellers and earn p units of money. Then, they become buyers again, and so on. Therefore, the stationary money holding distribution is degenerated at 0 and p. Let the populations be H_0 and H_1 respectively. Since the money supply is fixed at M, $p = M/H_1$.

In the market, each seller provides 1 unit of good, and each buyer purchases $p/p = 1$ unit. Hence, the market equilibrium condition of the good is

$$1 \cdot H_0 = 1 \cdot H_1.$$

Since the total population is 1, $H_0 = H_1 = 1/2$ and $p = 2M$. This is the unique equilibrium.

6.3 Bilateral Trade

Next, I assume that the Walrasian market incorporates a bilateral trade feature. Since Walrasian market basically allows multilateral trade, I just assume that the number of sellers must equal to the number of buyers in the market. Let $h(S)$ and $h(B)$ be the measure of agents who request to enter seller and buyer sides respectively. Then, each side only allows $\min\{h(S), h(B)\}$ agents to enter. The entrants are determined randomly.

Suppose that the price of the good is p, and the money holding distribution is degenerated at 0 and p with measures H_0 and H_1. Suppose $H_0 < H_1$ and $h \equiv H_0/H_1$. Sellers can always enter the market, but buyers get in with probability h. Under the assumptions, I show a continuum of equilibria. The agents solve

$$V(m) = \max\{V_s(m), V_b(m)\},$$

$$V_s(m) = \max_{q\in[0,1]} -c + \beta V(m + pq),$$

$$V_b(m) = h \cdot \max_{q\in[0,m/p]} \left[u \cdot \frac{q}{p} + \beta V(m - pq)\right] + (1 - h)\beta V(m).$$

The ex-ante value function can be rewritten as

$$V(m) = \max\left\{-c + \beta V(m + p),\ h\left[\frac{um}{p} + \beta V(0)\right] + (1 - h)\beta V(m)\right\}.$$

Then, the solution is

$$V(m) = \begin{cases} \frac{\beta hu-[1-\beta(1-h)]c}{1-\beta(1-h+\beta h)} & \text{if } m \leq \bar{m}, \\ \frac{hum}{p[1-\beta(1-h)]} + \beta\left(\frac{\beta hu-[1-\beta(1-h)]c}{1-\beta(1-h+\beta h)}\right) & \text{if } m \geq \bar{m}, \end{cases}$$

where $\bar{m} = \frac{p(\beta u-c)}{u(1+\beta)}$. The optimal choice is similar to the non-restriction case: choosing to be a seller if $m \leq \bar{m}$ and a buyer otherwise. The optimal decision is consistent with the money holding distribution.

On the equilibrium, each seller sales 1 unit of good. On the other side, each buyer holds p units of money and spend all at the price p. By the bilateral trade assumption, there are H_0 sellers and $H_0 = \min\{H_0, H_1\}$ buyers in the market. Therefore, the demand-supply equality is

$$1 \cdot H_0 = 1 \cdot H_0.$$

It holds for all $H_0 > 1/2$; hence, the money holding distribution is indterminate. Since the price must satisfy $H_1 p = [1 - H_0]p = M$, it is also indeterminate.

6.4 Relation to Kamiya and Shimizu (2013)

This result casts doubt on the conjecture by Kamiya and Shimizu (2013). They make an auction market and a competitive market model, and show the indeterminacy in the former and uniqueness in the latter. Since the auction market is centralized in some sense, they insist that the difference in price determination is important. The auction market uses game-theoretical equilibrium concept, so it is different from that of Walrasian market. However, the good is indivisible in their model. This means that the trade procedure in their auction market is restricted to bilateral trade, while that in Walrasian market is multilateral. Therefore, they might mistake the difference between bilateral and multilateral trades for the difference in the rules of the markets. This paper shows the indeterminacy in Walrasian market; hence, the result might dismiss their conjecture. In Kamiya and Shimizu (2013), they also conjecture that all-pay auction market shows the uniqueness. This result seems related to the multilateral trade market in this paper.

7 Logic of Indeterminacy and Uniqueness

In this section, to have a deeper understanding of the real indeterminacy, I follow Kamiya and Shimizu (2006) and analyze the abstract structure of the indeterminacy. Assume that there exist a continuum of single price equilibria as in Kamiya and Shimizu (2006). The distribution is degenerated at finite points $0, p, 2p, \ldots, Np$. Define $\mu = \{0, 1, \ldots, N\}$ as the set of states. A function $h(n)$ denotes the measure of people holding np units of money for $n = 0, 1, \ldots, N$. Kamiya and Shimizu (2006) show that search models of money with the real indeterminacy of stationary equilibria share the following system of equations.

Equations	Identities
$\sum_{n=0}^{N} h(n) = 1$	$\sum_{n=0}^{N} I_n = \sum_{n=0}^{N} O_n$
$\sum_{n=0}^{N} pnh(n) = M$	$\sum_{n=0}^{N} nI_n = \sum_{n=0}^{N} nO_n$
$I_0 = O_0$	Variables
$\vdots$	$h(0), \ldots, h(N)$
$I_N = O_N$	p

I_n and O_n are inflow and outflow in state n. There are $2 + (N + 1)$ equations, 2 identities and $(N + 1) + 1$ variables. Thus, the solution is indeterminate.

The key point is the identity $\sum_{n=0}^{N} nI_n = \sum_{n=0}^{N} nO_n$. Theorem 1 in Kamiya and Shimizu (2006) proves that the identity holds for broad search models of money in general. But the proof relies on the fact that population of agents moving from state $n \in \mu$ to $n + i \in \mu$ is equal to that of $m \in \mu$ to $m - i \in \mu$, where $i \in \mathbb{N}$. This assumption clearly depends on the bilateral trade procedure. Hence, the identity might be violated in multilateral trade models.

Example

Suppose the bilateral trade in Section 4 and multilateral trade in Section 5. Both models have single price equilibrium with $N = 1$. Under bilateral trade, $I_0 = I_1 = O_0 = O_1 = H_0 = \min\{H_0, H_1\}$. Hence, the identity is

$$0 \cdot H_0 + 1 \cdot H_0 = 0 \cdot H_0 + 1 \cdot H_0.$$

Therefore, it actually holds whatever H_0 is. However, under the multilateral trade, $I_0 = O_1 = H_1$ and $I_1 = O_0 = H_0$. Therefore, it is

$$0 \cdot H_1 + 1 \cdot H_0 = 0 \cdot H_0 + 1 \cdot H_1$$

which is not an identity. The unique solution is $H_0 = H_1 = 1/2$.

Therefore, the identity in Kamiya and Shimizu (2006) vanishes in the multilateral trade model. It ensures the uniquenss of the equilibrium.

8 Conclusion

This paper studies the real indeterminacy of stationary equilibrium in search theory of money. A price posting model with two alternative assumptions is considered. The equilibrium is indeterminate with bilateral trade assumption, while it is unique with multilateral trade. This finding suggests that bilateral trade procedure common in search theory might be the cause of the real indeterminacy.

The trade procedures in this paper are, in some sense, extremes for opposite directions. The bilateral trade assumption completely excludes agents who fail to make matches from the market. On the other hand, the multilateral trade allows all agents to be involved. An interesting extension will be a general case such as some agents match bilaterally, some have multilateral negotiations in small group, and some trade with many people, while some are exculded. I conjecture that the unique equilibrium might be derived. This is because the identity in Section 7 only holds with the pure bilateral trade assumption. In other words, it might be possible to show that the real indeterminacy happens only with exact bilateral cases.

Appendix

Proof of Proposition 1

In this proof, I will derive the equilibrium strategy x given the money holding distribution λ and the value function V. Next, the guess is verified by showing that λ and V are actually derived by the strategy x. To derive a subgame perfect equilibrium, I will derive a Nash equilibrium in each stage backward.

Stage 4

Given an offered price p and a money holding m, the problem is

$$\max_{q \in \mathbb{R}_+} uq + \beta V(m - pq), \quad \text{s.t. } pq \leq m, \; q \leq 1. \tag{A.1}$$

Lemma A.1. *If* $p \leq \left(\frac{1-\beta(1-h)}{h}\right) p^*$, *the buyer chooses* $q = 1$.

Proof. By Eq. (1), the slope of V is less than or equal to $\frac{hu}{p^*[1-\beta(1-h)]}$; hence, $q = 1$ maximizes the utlity. □

Since $\left(\frac{1-\beta(1-h)}{h}\right) \geq 1$, the buyer chooses $q = 1$ if $p = p^*$.

Stage 2 and 3

I consider Stage 2 and 3 at a time, and show that there exists a single price equilibrium with price p^*

Lemma A.2. *In Stage 2 and 3, all sellers post* p^*.

Proof. Consider a seller's strategy given that all other sellers post p^*. If the seller posts $p = p^*$, the seller can meet a buyer with probability 1 because the measure of sellers H_0 is less than that of buyers H_1. By Lemma 1, the seller will sell $q = 1$ unit of good and acquire p^* units of money.

Next, suppose that the seller offers $p > p^*$. Since all buyers have p^* units of money, the seller can get at most p^*. Since the cost is fixed at c, there is no incentive to offer $p > p^*$.

Finally, assume that the seller offers $p < p^*$. Since $q \leq 1$, the seller can earn at most $p < p^*$ units of money. The deviation is not profitable. □

Stage 1

Here, I calculate the discounted expected value of becoming a seller and a buyer. If an agent becomes a seller, she will post p^* and sell $q = 1$ with probability 1. It derives a discounted value

$$-c + \beta V(m + p^*). \tag{A.2}$$

Consider a case that the agent chooses to be a buyer. If $m < p^*$, she will meet a seller with probability h and spend all money holding. Hence, the discounted utility is

$$h\left(\frac{mu}{p^*} + \beta V(0)\right) + (1-h)V(m). \tag{A.3}$$

If $m \geq p^*$, she will spend at most p^* units of money. Then, the value is

$$h\left[u + \beta V\left(m - p^*\right)\right] + (1-h)V(m). \tag{A.4}$$

For later use, note that

$$V(p^*) - V(0) = \frac{hu - c}{1 + \beta h} \tag{A.5}$$

Lemma A.3. *An agent chooses to be a buyer if $m \geq p^*$.*

Proof. By Eq. (A.2) and Eq. (A.4), I will show the following:

$$-c + \beta V(m + p^*) \leq h\left[u + \beta V\left(m - p^*\right)\right] + (1-h)V(m)$$

$$\Leftrightarrow\ h\big[V(m + p^*) - V\left(m - p^*\right)\big] + (1-h)\big[V(m + p^*) - V(m)\big] \leq \frac{hu + c}{\beta}$$

By Eq. (1), I derive that $V(m + p^*) - V(m) \leq V(p^*) - V(0)$ and $V(m + p^*) - V\left(m - p^*\right) \leq V(2p^*) - V(0) \leq 2[V(p^*) - V(0)]$ for all $m \geq 0$. Hence, it is sufficient to show

$$(1+h)[V(p^*) - V(0)] \leq \frac{hu + c}{\beta}$$

By Eq. (A.5),

$$\Leftrightarrow\ \frac{(1+h)(hu + c)}{1 + \beta h} \leq \frac{hu + c}{\beta}.$$

Since $\beta(1+h) \leq 1 + \beta h$, it always holds. □

Lemma A.4. *An agent becomes a seller if $m < \bar{m}$ and a buyer if $\bar{m} \leq m \leq p^*$.*

Proof. Suppose $m < \bar{m}$ first. If the agent chooses to be a seller, the discounted value is, by Eq. (A.5),

$$-c + \beta V(m + p^*) = -c + \beta V(p^*) = -c + \beta\left(\frac{hu + c}{1 + \beta h} + V(0)\right)$$

$$= \frac{\beta(hu+c) - c(1+\beta h)}{1+\beta h} + \beta V(0) = \frac{\beta hu - c(1-\beta+\beta h)}{1+\beta h} + \beta V(0).$$

If she selects buyer, it is

$$h\left(\frac{mu}{p^*} + \beta V(0)\right) + \beta(1-h)V(m) = \frac{hmu}{p^*} + \beta V(0).$$

Therefore, I need to show

$$\frac{\beta hu - c(1-\beta+\beta h)}{1+\beta h} > \frac{hmu}{p^*}.$$

It can be rewritten as

$$\frac{p^*[\beta hu - c(1-\beta+\beta h)]}{hu(1+\beta h)} > m.$$

By Eq. (3), the LHS is exactly $\bar{m}$

Next, suppose a case that $m \geq \bar{m}$. If the agent chooses to become a seller, the expected utility is

$$
\begin{aligned}
&-c + \beta V(m+p^*) \\
&= -c + \beta V(p^*) + \beta\left(\frac{\beta h}{1-\beta(1-h)}\right)(m-\bar{m})\left(\frac{hu}{p^*[1-\beta(1-h)]}\right) \\
&= -c + \beta V(p^*) + \frac{\beta^2 h^2 u(m-\bar{m})}{p^*[1-\beta(1-h)]^2} \\
&= -c + \beta\left[\frac{hu+c}{1+\beta h} + V(0)\right] + \frac{\beta^2 h^2 u(m-\bar{m})}{p^*[1-\beta(1-h)]^2} \\
&= \frac{\beta hu - (1-\beta+\beta h)c}{1+\beta h} + \frac{\beta^2 h^2 u(m-\bar{m})}{p^*[1-\beta(1-h)]^2} + \beta V(0)
\end{aligned}
$$

If she decides to be a buyer, it is

$$
\begin{aligned}
&h\left(\frac{mu}{p^*} + \beta V(0)\right) + \beta(1-h)V(m) \\
&= h\left(\frac{mu}{p^*} + \beta V(0)\right) + \beta(1-h)\left[\frac{hu(m-\bar{m})}{p^*[1-\beta(1-h)]} + V(0)\right] \\
&= \frac{hmu}{p^*} + \beta(1-h)\frac{hu(m-\bar{m})}{p^*[1-\beta(1-h)]} + \beta V(0).
\end{aligned}
$$

I need to show

$$-c + \beta V(m+p^*) \leq h\left(\frac{mu}{p^*} + \beta V(0)\right) + \beta(1-h)V(m),$$

which is rewritten as

$$
\begin{aligned}
&\frac{\beta hu-(1-\beta+\beta h)c}{1+\beta h}+\frac{\beta^2h^2u(m-\bar{m})}{p^*[1-\beta(1-h)]^2}\leq\frac{hmu}{p^*}+\beta(1-h)\frac{hu(m-\bar{m})}{p^*[1-\beta(1-h)]}\\
\Leftrightarrow\ &\frac{\beta hu-(1-\beta+\beta h)c}{1+\beta h}+\frac{\beta hu(m-\bar{m})}{p^*[1-\beta(1-h)]}\left[\frac{\beta h}{1-\beta(1-h)}-(1-h)\right]\leq\frac{hmu}{p^*}\\
\Leftrightarrow\ &\frac{p^*[\beta hu-(1-\beta+\beta h)c]}{hu(1+\beta h)}+\frac{\beta(m-\bar{m})}{1-\beta(1-h)}\left[\frac{(1-h)(1-\beta)-h^2\beta}{1-\beta(1-h)}\right]\leq m\\
\Leftrightarrow\ &\bar{m}+\frac{\beta(m-\bar{m})}{1-\beta(1-h)}\left[\frac{(1-h)(1-\beta)-h^2\beta}{1-\beta(1-h)}\right]\leq m\\
\Leftrightarrow\ &\bar{m}\left[1-\frac{\beta(1-h)(1-\beta)-h^2\beta^2}{[1-\beta(1-h)]^2}\right]\leq m\left[1-\frac{\beta(1-h)(1-\beta)-h^2\beta^2}{[1-\beta(1-h)]^2}\right]\\
\Leftrightarrow\ &\bar{m}\leq m.
\end{aligned}
$$

□

By Lemma 4, $\bar{m}$ is the cutoff.

Lemma A.5. *The Equilibrium Guess is verified*

Proof. I will show that the money holding distribution is stationary, and also derive the value function from the equilibrium strategy.

The former is clear. On the equilibrium, H_0 agents hold no money and choose to be sellers, and H_1 agents hold p^* and become buyers. The buyers spend all money holding, then H_1 agents' money holdings become 0. At the same time, H_1 in H_0 sellers acquire M/H_1 units of money. Therefore, the money holding distribution is stationary.

Next I derive the value function. First, suppose an agent holding $m<\bar{m}$. She becomes a seller, then the expected return is

$$
\begin{aligned}
&-c+\beta V(m+p^*)=-c+\beta V(p^*)\\
&=-c+\frac{\beta hu}{1-\beta(1-h)}+\left(\frac{\beta^2h}{1-\beta(1-h)}\right)V(0)\\
&=-c+\frac{\beta hu}{1-\beta(1-h)}+\left(\frac{\beta^2h}{1-\beta(1-h)}\right)\frac{\beta hu-(1-\beta+\beta h)c}{(1-\beta)(1+\beta h)}\\
&=\left(\frac{(1-\beta)(1+\beta h)+\beta^2h}{[1-\beta(1-h)](1-\beta)(1+\beta h)}\right)\beta hu-\left(\frac{\beta^2h}{(1-\beta)(1+\beta h)}+1\right)c\\
&=\frac{\beta hu-(1-\beta+\beta h)c}{(1-\beta)(1+\beta h)}. \qquad \text{(A.6)}
\end{aligned}
$$

By Eq. (2), Eq. (A.6) is exactly $V(0)$. Next, if $p^* > m \geq \bar{m}$, the agent becomes a buyer. Since the agent trades with probability h, the discounted utility is

$$h\left(\frac{m}{p^*}u + \beta V(0)\right) + \beta(1-h)V(m) \tag{A.7}$$

This coincides with Eq. (4).

Finally, suppose $m \geq p^*$. The agent becomes a buyer and the expected utility is

$$h[u + \beta V(m - p^*)] + \beta(1-h)V(m). \tag{A.8}$$

Eq. (A.8) coincides with Eq. (6). □

Proof of Property 4

By Property 3, all sellers post the single-price p on equilibrium. By the assumption of the multilateral trade, the revenue of each seller is the same. Let α denote the revenue.

By the definition of the equilibrium, there exists m such that $V(0) < V(m)$. It implies that an agent with sufficient amount of money becomes a buyer. Suppose that an agent with $\tilde{m}$ units of money chooses to be a buyer in equilibrium. By Property 2, the agent spends whole money holding. The optimality of this agent requires the condition

$$u\frac{\tilde{m}}{p} + \beta V(0) \geq -c + \beta V(\tilde{m} + \alpha).$$

Now, I will show that, for all $m' > \tilde{m}$, an agent who holds m' units of money becomes a buyer. Consider an agent holding $m' + \alpha$. Since $V(m' + \alpha) \geq V(\tilde{m})$, the agent spends money in some future period. Suppose that an agent holding $m' + \alpha$ optimally use all money holding in T periods ahead. In addition, define that $V^\tau(m)$ is the discounted utility when an agent will be a seller in $t, t+1, \ldots, t+\tau-1$ and will be a buyer and spend whole amount of money in $t+\tau$. In this case, $V(m'+\alpha) = V^T(m'+\alpha)$ because this strategy is optimal when the money holding is $m' + \alpha$. Therefore,

$$V(m' + \alpha) = V^T(\tilde{m} + \alpha) + \beta^T u\frac{(m' - \tilde{m})}{p} \leq V(\tilde{m} + \alpha) + \beta^T u\frac{(m' - \tilde{m})}{p}$$

where the first equation holds because the only difference between $V(m' + \alpha)$ and $V(\tilde{m} + \alpha)$ is the amount of spending in $t + T$, and the second inequality is due to the optimality. Since $\beta^T < \beta$, $V(m' + \alpha) - V(\tilde{m} + \alpha) < \beta u(m' - \tilde{m})/p^*$. Then, by the optimality at $\tilde{m}$,

$$u\frac{\tilde{m}}{p^*} + \beta V(0) \geq -c + \beta V(\tilde{m} + \alpha) \;\Rightarrow\; u\frac{m'}{p^*} + \beta V(0) > -c + \beta V(m' + \alpha).$$

This implies that an agent holding m' chooses to be a buyer immediately. It means that all agents hold more than or equal to $\tilde{m}$ become buyers.

Define $\bar{m}$ be the infimum of such $\tilde{m}$, i.e., an agent becomes a buyer if $m \geq \bar{m}$ and be a seller otherwise. Since all agents holding $m' \geq m$ become buyers and spend all money, $V(m') - V(m) = u(m' - m)/p^*$. Therefore, the value function is linear in $[\bar{m}, \infty)$.

Proof of Proposition 3

First, I show $p^* = 2M > \bar{m}$. By the definition, the discounted value of being a buyer or seller is indifferent when an agent holds $\bar{m}$. Hence,

$$V(\bar{m}) = u \cdot \frac{\bar{m}}{2M} + \beta V(0) = -c + \beta \left[u \cdot \frac{\bar{m} + 2M}{2M} + \beta V(0) \right]. \tag{A.9}$$

An agent holding exactly $\bar{m}$ unit of money and 0 unit of money become sellers this period and spend money next period. Then,

$$V(\bar{m}) - V(0) = \beta \cdot u \cdot \frac{\bar{m}}{2M}, \tag{A.10}$$

because the only difference is the amount of money spended in the next period. By Eq. (A.9) and Eq. (A.10), it is derived that

$$V(0) = \frac{u\bar{m}}{2M},$$

$$\bar{m} = \frac{2M(\beta u - c)}{u(1-\beta)(1+\beta)}.$$

The market participation condition $V(0) \geq 0$ is derived by a premise $\beta u - c \geq 0$. Besides $c > (\beta + \beta^2 - 1)u$ ensures $\bar{m} < 2M = p^*$.

Next, I check the optimality of (9). By the above argument, the value function is derived as

$$V(m) = \begin{cases} V(0) + \frac{\beta u m}{p^*} = \frac{u\bar{m}}{2M} + \frac{\beta u m}{2M} & \text{if } m \leq \bar{m} \\ V(\bar{m}) + \frac{um}{p^*} = \frac{u\bar{m}}{2M} + \frac{\beta u \bar{m}}{2M} + \frac{um}{2M} & \text{if } m > \bar{m} \end{cases}$$

On the equilibrium, $h(S_p) = 1/2$, $Q_p = 1/2$, $p^* = 2M$. Then the optimization problem (8) is rewritten as

$$\max_{q \in \mathbb{R}_+} \; uq + \beta V(m - 2Mq) \quad \text{s.t. } 2Mq \leq m.$$

Because the slope of the value function is $\beta u/2M$ or $u/2M$, the objective function is strictly increasing in q. Threfore, $q = m/2M$ is the solution.

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